Set-membership estimation for linear time-varying descriptor systems ⋆

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Abstract

This paper considers the problem of set-membership estimation for discrete-time linear time-varying descriptor systems subject to unknown but bounded disturbance and noise. We propose a set-membership estimation method based on a descriptor system observer and a zonotopic estimator of the observer error bounds. The observer parameters are optimized in order to minimize the sizes of the zonotopes enclosing all admissible state trajectories. Finally, two simulation results are provided to demonstrate the effectiveness of the proposed method.

Key words: Set-membership estimation; Linear time-varying; Descriptor systems; Zonotopes.

1 Introduction

State estimation is important in control theory and applications. However, the ubiquitous uncertainties influence the performance of state estimation. Many existing state estimation methods assume that the uncertainties have known probability distribution functions, which may not be true in practice (Alamo, Bravo & Camacho, 2005). Based on a general assumption that the uncertainties are unknown but bounded, set-membership estimation requires no information about the probability distributions of the uncertainties. Set-membership estimation methods have been applied to various fields such as bioprocess monitoring (Gouzé, Rapaport & Hadj-Sadok, 2000; Moisan, Bernard & Gouzé, 2009), robust control (Canale, Fagiano & Milanese, 2009; Efimov, Raïssi & Zolghadri, 2013b; Zhang, Yang, Han & Zhu, 2019) and fault diagnosis (Xu, Tan, Wang, Wang, Liang & Yuan, 2019). Different kinds of geometrical sets such as ellipsoids (Fogel & Huang, 1982; Chernousko, 2005), polytopes (Shamma & Tu, 1999; Blanchini & Miani, 2008) and intervals (Raïssi, Efimov & Zolghadri, 2012; Efimov, Raïssi, Chebotarev & Zolghadri, 2013a; Wang, Lim & Shen, 2018b) have been used to design set-membership estimation methods. Among these sets, zonotopes can achieve a good tradeoff between estimation accuracy and computation complexity. Recently, the set-membership estimation methods based on zonotopes have received much attention (Le, Stoica, Alamo, Camacho & Dumur, 2013; Combastel, 2015; Scott, Raimondo, Marseglia & Braatz, 2016; Tang, Wang, Wang, Raïssi & Shen, 2019).

Most works on set-membership estimation focus on regular systems. Descriptor systems are more general and can describe many practical systems. Traditional state estimation methods for descriptor systems have been studied extensively. However, only few results study set-membership estimation for descriptor systems. Wang, Puig & Cembrano (2018a) and Wang, Olaru, Valmorbida, Puig & Cembrano (2019a) studied set-membership estimation methods for descriptor systems based on zonotopes. However, Wang et al. (2018a) and Wang et al. (2019a) only considered time-invariant systems.

In this paper, we propose a new set-membership estimation method for discrete-time linear time-varying (LTV) descriptor systems subject to unknown but bounded disturbances and measurement noises. The main contributions of this paper reside in the following aspects:

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1) A novel set-membership estimation method for discrete-time LTV descriptor systems based on a simple observer structure, which is easy to design and implement.
2) The optimization of all the parameters of the proposed set-membership estimator, with a closed-form solution.
3) A rigorous proof of the optimality minimizing the zonotope size criterion, not simply relying on the first order condition.

Set-membership estimation for LTV descriptor systems has been studied in Wang, Wang, Puig & Cembrano (2019b), but the estimator presented in Wang et al. (2019b) involves coupled parameters, which are only partly optimized. It is also limited to descriptor systems with time-invariant output equations. In contrast, the proposed method is applicable to fully LTV descriptor systems and can optimize all the estimator parameters. Moreover, sufficient optimality conditions are established for the proposed estimator, whereas in some similar results, only the necessary first order condition is considered, like in Combastel (2015).

Notations: $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote the n and $m \times n$ dimensional Euclidean spaces, respectively. I_n denotes the identity matrix with dimensions of $n \times n$. For a matrix $A \in \mathbb{R}^{m\times n}$, A^{T} denotes its transpose and $A^{\dagger}$ represents its pseudo-inverse. $\mathbf{vec}(A)$ denotes the vectorization of A by concatenating the columns of A into a single column vector. $\sigma_{\min}(A)$ denotes the smallest singular value of A. For a square matrix A, $\mathrm{tr}(A)$ is its trace. For two matrices X and Y, $X \otimes Y$ denotes their Kronecker product. In this paper, the symbols $\geq$, $>$, $\leq$, $<$ and the absolute value operator $|\cdot|$ should be understood element-wise. $P \succ 0$ ($P \prec 0$) indicates that P is a positive (negative) definite matrix. $P \succeq 0$ ($P \preceq 0$) indicates that P is a positive (negative) semidefinite matrix.

2 Preliminaries

Definition 1 *An m-order zonotope $\mathcal{Z} \subset \mathbb{R}^n$ is the affine transformation of a hypercube $\mathbf{B}^m = [-1, 1]^m$:*

$$\mathcal{Z} = p \oplus H\mathbf{B}^m = \{p + Hz : z \in \mathbf{B}^m\},$$

where $\oplus$ denotes the Minkowski sum, $p \in \mathbb{R}^n$ is the center of $\mathcal{Z}$, and $H \in \mathbb{R}^{n\times m}$ is called the generator matrix of $\mathcal{Z}$. For simplicity, we also denote $\mathcal{Z} = \langle p, H\rangle$.

Property 1 *(Combastel, 2015) For zonotopes, the following properties hold:*

$$\langle p_1, H_1\rangle \oplus \langle p_2, H_2\rangle = \langle p_1 + p_2, [H_1 \quad H_2]\rangle, \tag{1a}$$
$$L\langle p, H\rangle = \langle Lp, LH\rangle, \tag{1b}$$
$$\langle p, H\rangle \subseteq \langle p, \mathrm{rs}(H)\rangle, \tag{1c}$$

where p, p_1, p_2 $\in$ $\mathbb{R}^n$, H, H_1, H_2 $\in$ $\mathbb{R}^{n\times m}$ and $L \in \mathbb{R}^{l\times n}$. $\mathrm{rs}(H) = \mathrm{diag}([h_1, \dots, h_n])$ where $h_i = \sum_{j=1}^{m} |H_{i,j}|$, $i = 1, \dots, n$. Note that $\langle p, \mathrm{rs}(H)\rangle$ is also a box, i.e., an interval vector, which represents a box outer bound of $\langle p, H\rangle$.

A high-order zonotope can be bounded by a lower one via the reduction operation based on the following property.

Property 2 *(Alamo et al., 2005) Given the zonotope $\mathcal{Z} = p \oplus H\mathbf{B}^q \subset \mathbb{R}^n$ and a chosen integer s ($n < s < q$), denote the matrix obtained by reordering the columns of H in decreasing Euclidean norm as $\hat{H}$. Then $\mathcal{Z} \subset \langle p, \mathscr{R}_s(H)\rangle$, where $\mathscr{R}_s(H) = \begin{bmatrix} H^a & H^b \end{bmatrix}$, H^a is composed of the first $s - n$ columns of $\hat{H}$ and H^b is a diagonal matrix with*

$$H^b_{i,i} = \sum_{j=s-n+1}^{q} |\hat{H}_{i,j}|, i = 1, \dots, n.$$

Property 3 *(Golub & Van Loan, 1996) For matrices X, A, B, C with appropriate dimensions, the following equations hold:*

$$\mathrm{tr}(A) = \mathrm{tr}(A^{\mathrm{T}}), \tag{2a}$$
$$\frac{\partial}{\partial X}\mathrm{tr}(AX^{\mathrm{T}}B) = BA, \tag{2b}$$
$$\frac{\partial}{\partial X}\mathrm{tr}(AXBX^{\mathrm{T}}C) = A^{\mathrm{T}}C^{\mathrm{T}}XB^{\mathrm{T}} + CAXB. \tag{2c}$$

Lemma 1 *(Wang et al., 2018b) Given matrices $\mathcal{W} \in \mathbb{R}^{p\times n}$ and $\mathcal{Y} \in \mathbb{R}^{m\times n}$, if $\mathrm{rank}(\mathcal{W}) = n$, then the general solution to $\mathcal{X}\mathcal{W} = \mathcal{Y}$ exists and is given by*

$$\mathcal{X} = \mathcal{Y}\mathcal{W}^{\dagger} + \mathcal{S}(I_p - \mathcal{W}\mathcal{W}^{\dagger})$$

where $\mathcal{S} \in \mathbb{R}^{m\times p}$ is an arbitrary matrix.

3 Problem statement

Consider the following system

$$\begin{cases} E_k x_k = A_k x_{k-1} + B_k u_{k-1} + w_{k-1} \\ y_k = C_k x_k + v_k \end{cases} \tag{3}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_{k-1} \in \mathbb{R}^{n_u}$ and $y_k \in \mathbb{R}^{n_y}$ are the vectors of state, input and measurement output, respectively. $E_k \in \mathbb{R}^{n_x\times n_x}$, $A_k \in \mathbb{R}^{n_x\times n_x}$, $B_k \in \mathbb{R}^{n_x\times n_u}$ and $C_k \in \mathbb{R}^{n_y\times n_x}$ are known matrices. E_k may be singular. $w_{k-1} \in \mathbb{R}^{n_x}$ and $v_k \in \mathbb{R}^{n_y}$ denote the unknown disturbance and the measurement noise, respectively.

For system (3), we have the following assumptions.

Assumption 1. x_0, w_k and v_k are unknown but bounded as follows:

$$x_0 \in \langle p_0, H_0 \rangle, \quad w_k \in \langle 0, D_k \rangle, \quad v_k \in \langle 0, F_k \rangle, \quad \forall k \geq 0,$$

where $p_0 \in \mathbb{R}^{n_x}$, $H_0 \in \mathbb{R}^{n_x \times n_x}$, $D_k \in \mathbb{R}^{n_x \times n_x}$ and $F_k \in \mathbb{R}^{n_y \times n_y}$ are known vector and matrices.

Assumption 2. We assume that

$$\sigma_{\min}(D_k) \geq \epsilon_1, \quad \sigma_{\min}(F_k) \geq \epsilon_2, \quad \forall k \geq 0 \tag{4}$$

where ϵ_1 and ϵ_2 are positive constants.

Assumption 3. The parameters of system (3) are assumed to satisfy the following condition:

$$\operatorname{rank} \begin{bmatrix} E_k \\ C_k \end{bmatrix} = n_x, \quad \forall k \geq 0. \tag{5}$$

Remark 1. Assumption 3 is commonly used in descriptor system estimation (e.g. Nikoukhah, Willsky & Levy, 1990; Hamdi, Rodrigues, Mechmeche & Braiek, 2012; Zhang, Swain & Nguang, 2014).

In this paper, we aim to develop an optimal set-membership estimation method for (3) to obtain the minimized zonotope sets that can tightly enclose all admissible state trajectories. In addition, we can further obtain the upper and lower bounds of x_k, i.e., an interval vector $[\underline{x}_k, \overline{x}_k]$ satisfying $\underline{x}_k \leq x_k \leq \overline{x}_k$.

4 Main results

4.1 Set-membership estimator

We propose a set-membership estimator for system (3) to obtain zonotopes that can enclose possible state trajectories. The proposed set-membership estimator will consist of two parts: a single state trajectory estimator and a bound estimator of the estimation error of this trajectory estimator. The first part is based on the observer structure proposed in Wang, Shen, Zhang & Wang (2012), which is in the form of

$$\begin{aligned} \hat{x}_k = T_k A_k \hat{x}_{k-1} + T_k B_k u_{k-1} + N_k y_k \\ + L_k (y_{k-1} - C_{k-1} \hat{x}_{k-1}), \end{aligned} \tag{6}$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is the single trajectory estimation, $T_k \in \mathbb{R}^{n_x \times n_x}$, $N_k \in \mathbb{R}^{n_x \times n_y}$ and $L_k \in \mathbb{R}^{n_x \times n_y}$ are the matrices to be designed. T_k and N_k should satisfy

$$T_k E_k + N_k C_k = I_{n_x}. \tag{7}$$

Remark 2. According to Lemma 1, if Assumption 3 holds, then there exist T_k and N_k such that (7) holds.

We use observer (6) to generate the single trajectory estimation. Then, the following theorem is used to obtain the set-membership estimation.

Theorem 1 *Given matrices T_k, N_k and L_k, if $x_{k-1} \in \langle \hat{x}_{k-1}, H_{k-1} \rangle \subset \mathbb{R}^{n_x}$, then $x_k \in \langle \hat{x}_k, H_k \rangle$, where $\hat{x}_k$ is obtained from (6) and H_k satisfies*

$$H_k = \left[(T_k A_k - L_k C_{k-1}) \widetilde{H}_{k-1} \ G_k \right], \tag{8}$$

where

$$\begin{aligned} G_k &= \left[T_k D_{k-1} \ -L_k F_{k-1} \ -N_k F_k \right], \\ \widetilde{H}_{k-1} &= \begin{cases} \mathscr{R}_s(H_{k-1}), & q > s; \\ H_{k-1}, & q \leq s. \end{cases} \end{aligned} \tag{9}$$

Herein, q is the number of columns of H_{k-1} and s is a fixed integer.

PROOF. By combining (3) with (7), we have

$$\begin{aligned} x_k = T_k A_k x_{k-1} + T_k B_k u_{k-1} + T_k w_{k-1} + N_k y_k \\ - N_k v_k. \end{aligned} \tag{10}$$

Define the estimation error as $e_k = x_k - \hat{x}_k$. Since $x_{k-1} \in \langle \hat{x}_{k-1}, H_{k-1} \rangle$, it follows that $e_{k-1} \in \langle 0, H_{k-1} \rangle$.

According to Property 2, (9) implies $\langle 0, H_{k-1} \rangle \subseteq \langle 0, \widetilde{H}_{k-1} \rangle$. Then, we have $e_{k-1} \in \langle 0, \widetilde{H}_{k-1} \rangle$.

Subtracting (6) from (10) yields

$$\begin{aligned} e_k = (T_k A_k - L_k C_{k-1}) e_{k-1} + T_k w_{k-1} - L_k v_{k-1} \\ - N_k v_k. \end{aligned} \tag{11}$$

According to Assumption 1, (11) implies

$$\begin{aligned} e_k \in (T_k A_k - L_k C_{k-1}) \langle 0, \widetilde{H}_{k-1} \rangle \oplus T_k \langle 0, D_{k-1} \rangle \\ \oplus (-L_k \langle 0, F_{k-1} \rangle) \oplus (-N_k \langle 0, F_k \rangle) \end{aligned}$$

Then, by using Property 1, we have $e_k \in \langle 0, H_k \rangle$.

Since $x_k = \hat{x}_k + e_k$, it follows that $x_k \in \langle \hat{x}_k, H_k \rangle$. □

Remark 3. According to the property in (1c), we can obtain the upper and lower bounds of x_k as follows:

$$\begin{cases} \overline{x}_k = \hat{x}_k + \operatorname{rs}(H_k), \\ \underline{x}_k = \hat{x}_k - \operatorname{rs}(H_k). \end{cases}$$

4.2 Optimization design

To obtain accurate estimation, the matrices T_k, N_k and L_k should be designed to minimize the size of the constructed zonotope set, $\langle \hat{x}_k, H_k \rangle$. In this paper, we chose

$$J_k(T_k, N_k, L_k) = \|H_k(T_k, N_k, L_k)\|_F^2 = \mathrm{tr}(H_k^{\mathrm{T}} H_k) \tag{12}$$

as the size criterion of $\langle \hat{x}_k, H_k \rangle$. It can be interpreted as the size of the zonotope generator segments, as mentioned in Alamo et al. (2005) and Combastel (2015), or more precisely, the sum of the squares of the generator segment half lengths. It is chosen in this paper due to its simplicity in analysis and implementation.

From (8), J_k is a quadratic form in terms of T_k, N_k and L_k. In addition, T_k and N_k are required to satisfy the equality constraint (7). Therefore, we aim to solve a constrained quadratic optimization problem as follows:

$$\min_{T_k E_k + N_k C_k = I_{n_x}} J_k(T_k, N_k, L_k). \tag{13}$$

According to Lemma 1, T_k and N_k can be chosen as

$$\begin{aligned} T_k &= \Theta_k^{\dagger} \alpha_1 + S_k \Omega_k \alpha_1, \\ N_k &= \Theta_k^{\dagger} \alpha_2 + S_k \Omega_k \alpha_2, \end{aligned} \tag{14}$$

where

$$\Theta_k = \begin{bmatrix} E_k \\ C_k \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} I_{n_x}, \\ 0 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 \\ I_{n_y} \end{bmatrix},$$
$$\Omega_k = I_{n_x+n_y} - \Theta_k \Theta_k^{\dagger}.$$

Note that it is difficult to minimize the size criterion J_k in terms of S_k, because the matrix Ω_k is always rank deficient. In Wang et al. (2019b), S_k is chosen based on personal experience. In fact, if S_k is given, we can obtain the optimal L_k to minimize the size criterion.

Theorem 2 *Given any matrix pair T_k and N_k of appropriate dimensions satisfying (7) (possibly chosen following (14)), then the L_k obtained from (15) is the optimal one minimizing the size criterion J_k.*

$$\begin{aligned} V_{k-1} &= C_{k-1} P_{k-1} C_{k-1}^{\mathrm{T}} + R_{k-1}, \\ L_k &= T_k A_k P_{k-1} C_{k-1}^{\mathrm{T}} {V_{k-1}}^{-1}, \end{aligned} \tag{15}$$

where

$$P_{k-1} = \widetilde{H}_{k-1} \widetilde{H}_{k-1}^{\mathrm{T}}, \quad R_{k-1} = F_{k-1} F_{k-1}^{\mathrm{T}}. \tag{16}$$

PROOF. From (8), we have

$$\begin{aligned} J_k = {} & \mathrm{tr}\big((T_k A_k - L_k C_{k-1}) P_{k-1} (T_k A_k - L_k C_{k-1})^{\mathrm{T}}\big) \\ & + \mathrm{tr}(T_k Q_{k-1} T_k^{\mathrm{T}}) + \mathrm{tr}(L_k R_{k-1} L_k^{\mathrm{T}}) \\ & + \mathrm{tr}(N_k R_k N_k^{\mathrm{T}}), \end{aligned}$$

where $Q_{k-1} = D_{k-1} D_{k-1}^{\mathrm{T}}$. By using the matrix calculus properties in (2), we can obtain the partial derivative of J_k for L_k as follows:

$$\frac{\partial J_k}{\partial L_k} = 2L_k (C_{k-1} P_{k-1} C_{k-1}^{\mathrm{T}} + R_{k-1}) - 2 T_k A_k P_{k-1} C_{k-1}^{\mathrm{T}}.$$

The L_k obtained from (15) satisfies $\frac{\partial J}{\partial L_k} = 0$, which means that it is a stationary point.

From (4) and (16), we have $R_{k-1} \succ 0$. Since $V_{k-1} \succeq R_{k-1}$, it follows that V_{k-1} is invertible. Therefore, the L_k obtained from (15) is the unique stationary point of J_k. In addition, J_k is a quadratic function of L_k and by definition $J_k \geq 0$ for any L_k, then J_k is a convex function of L_k. Therefore, the L_k obtained from (15) is the unique minimum point of J_k. $\square$

Although T_k and N_k can be obtained from (14), it is difficult to find the optimal S_k minimizing the size criterion J_k. A freely chosen S_k may cause large conservatism.

Instead of optimizing T_k and N_k with the parametrization in terms of S_k, we propose the following theorem directly optimizing T_k, N_k and L_k to minimize the size criterion J_k, under the equality constraint (7).

Theorem 3 *The constrained optimization problem (13) has a unique optimum in terms of T_k, N_k and L_k, which is explicitly obtained from*

$$V_{k-1} = C_{k-1} P_{k-1} C_{k-1}^{\mathrm{T}} + R_{k-1}, \tag{17a}$$
$$M_k = A_k P_{k-1} C_{k-1}^{\mathrm{T}} {V_{k-1}}^{-1} C_{k-1} P_{k-1} A_k^{\mathrm{T}}, \tag{17b}$$
$$W_k = A_k P_{k-1} A_k^{\mathrm{T}} + Q_{k-1}, \tag{17c}$$
$$O_k = 2(M_k - W_k), \tag{17d}$$
$$\Psi_k = -2R_k, \tag{17e}$$
$$\Lambda_k = (E_k^{\mathrm{T}} {O_k}^{-1} E_k + C_k^{\mathrm{T}} {\Psi_k}^{-1} C_k)^{-1}, \tag{17f}$$
$$T_k = \Lambda_k E_k^{\mathrm{T}} {O_k}^{-1}, \tag{17g}$$
$$N_k = \Lambda_k C_k^{\mathrm{T}} {\Psi_k}^{-1}, \tag{17h}$$
$$L_k = T_k A_k P_{k-1} C_{k-1}^{\mathrm{T}} {V_{k-1}}^{-1}. \tag{17i}$$

PROOF. To prove that the solution (T_k, N_k, L_k) obtained from (17) is the unique optimum of the constrained optimization problem (13), usually the first order and the second order optimality conditions should be checked. However, this is a non-trivial task due to the fact that the constraint (7) and the size criterion

J_k are formulated in terms of parameter *matrices*, instead of the *vector* form in standard quadratic optimization problems. To overcome this difficulty, we conduct our proof in two steps. First, we prove that the solution $(T_k, N_k, L_k, \Lambda_k)$ obtained from (17) is the unique stationary point of the Lagrange function of the constrained optimization problem. Second, we verify that the constrained optimization problem (13) satisfies the condition ensuring that the unique stationary point is the unique optimum of the considered problem.

First, define the following Lagrange function:

$$Z_k(T_k, N_k, L_k, \Lambda_k) = J_k(T_k, N_k, L_k) + \mathrm{tr}(\Lambda_k \Phi_k),$$

where $\Lambda_k \in \mathbb{R}^{n_x \times n_x}$ and $\Phi_k = E_k^{\mathrm{T}} T_k^{\mathrm{T}} + C_k^{\mathrm{T}} N_k^{\mathrm{T}} - I_{n_x}$. According to the matrix calculus properties in (2), the partial derivatives of Z_k for T_k, N_k, L_k and Λ_k are

$$\begin{aligned}
\frac{\partial Z_k}{\partial T_k} &= 2T_k W_k - 2L_k C_{k-1} P_{k-1} A_k^{\mathrm{T}} + \Lambda_k E_k^{\mathrm{T}}, \\
\frac{\partial Z_k}{\partial N_k} &= 2N_k R_k + \Lambda_k C_k^{\mathrm{T}}, \\
\frac{\partial Z_k}{\partial L_k} &= 2L_k V_{k-1} - 2T_k A_k P_{k-1} C_{k-1}^{\mathrm{T}}, \\
\frac{\partial Z_k}{\partial \Lambda_k} &= T_k E_k + N_k C_k - I_{n_x}.
\end{aligned}$$

Let all the partial derivatives equal to 0, we have:

$$\begin{cases} X_k \Sigma_k + \Lambda_k \Pi_k^{\mathrm{T}} = 0, & \text{(18a)} \\ X_k \Pi_k = I_{n_x}, & \text{(18b)} \end{cases}$$

where

$$X_k = \begin{bmatrix} T_k & N_k & L_k \end{bmatrix}, \quad \Pi_k = \begin{bmatrix} E_k \\ C_k \\ 0 \end{bmatrix},$$
$$\Sigma_k = 2 \begin{bmatrix} W_k & 0 & -A_k P_{k-1} C_{k-1}^{\mathrm{T}} \\ 0 & R_k & 0 \\ -C_{k-1} P_{k-1} A_k^{\mathrm{T}} & 0 & V_{k-1} \end{bmatrix}. \tag{19}$$

From (17a) and (17c), the matrix Σ_k can be rewritten as

$$\Sigma_k = \Xi_k + \Gamma_k, \tag{20}$$

where

$$\Xi_k = 2 \begin{bmatrix} Q_{k-1} & 0 & 0 \\ 0 & R_k & 0 \\ 0 & 0 & R_{k-1} \end{bmatrix},$$
$$\Gamma_k = 2 \begin{bmatrix} A_k \\ 0 \\ -C_{k-1} \end{bmatrix} P_{k-1} \begin{bmatrix} A_k \\ 0 \\ -C_{k-1} \end{bmatrix}^{\mathrm{T}}.$$

From (4), we have $Q_k \succ 0$ and $R_k \succ 0$ for any $k \geq 0$. It follows that $\Xi_k \succ 0$. In addition, $\Gamma_k \succeq 0$ since $P_{k-1} \succeq 0$. Therefore, from (20), we have $\Sigma_k \succ 0$, which implies that Σ_k is invertible. Then from (18a), we have

$$X_k = -\Lambda_k \Pi_k^{\mathrm{T}} \Sigma_k^{-1}. \tag{21}$$

Substituting it into (18b) yields

$$\Lambda_k(-\Pi_k^{\mathrm{T}} \Sigma_k^{-1} \Pi_k) = I_{n_x}. \tag{22}$$

From (5), we have $\mathrm{rank}\Pi_\mathrm{k} = \mathrm{n_x}$. Then, Π^{T} is row full rank. Since Σ_k^{-1} is invertible, it follows that $-\Pi_k^{\mathrm{T}} \Sigma_k^{-1} \Pi_k$ is invertible. Therefore, Λ_k can be uniquely determined by (22). Then, $X_k = \begin{bmatrix} T_k & N_k & L_k \end{bmatrix}$ is uniquely determined by (21). Therefore, (18) has one unique solution. It is easy to check that $(T_k, N_k, L_k, \Lambda_k)$ obtained from (17) is the solution to (18). Therefore, $(T_k, N_k, L_k, \Lambda_k)$ obtained from (17) is the unique stationary point of the Lagrange function.

Next, we will verify that the unique stationary point is the unique optimum of the considered problem.

Define $x = \mathbf{vec}(X_k)$. The size criterion J_k defined in (12) is a quadratic form in x (a homogeneous polynomial containing only second degree terms of x). Moreover, this quadratic form is positive semidefinite, since it can never be negative for any value of x.

From the constraint (7), we have $X_k \Pi_k = I_{n_x}$, which is equivalent to

$$\mathcal{A}x = \mathbf{vec}(I_{n_x}),$$

where $\mathcal{A} = \Pi_k^{\mathrm{T}} \otimes I_{n_x}$. Since Π_k^{T} is row full rank, it follows that $\mathcal{A}$ is row full rank. Then, according to Markowitz (1956), the solution obtained from (17) is the unique optimum of the considered problem. □

Remark 4. Assumption 2 has been introduced to establish, in Theorem 3, the optimality of the proposed solution. This assumption can be relaxed if the optimal solution is simply deduced by the first order optimality condition, like in Combastel (2015), which is not sufficient to ensure the optimality. In fact, the optimality proof of Theorem 3 requires that the matrix Σ_k is invertible, which is a weaker condition than the conditions formulated in (4). However, in practice, it is more difficult to check the large matrix Σ_k defined in (19), and easier to check the matrices D_k and F_k. Assumption 2 means that uncertainties exist in all the directions of the state space and of the output space. In practice, if there is no uncertainty in some directions, a simple solution is to slightly modify D_k or F_k so that (4) is satisfied with a small value of ϵ_1 or ϵ_2. This approach is often used for parameter estimation with methods related to the Kalman filter. Alternatively, for a more accurate solution, the dimension of the state equation or of the output equation could be reduced so that Assumption 2 becomes true, but this approach still requires non-trivial work in future studies.

5 Simulation results

A numerical example is used to demonstrate the performance of the proposed method, which has parameters as follows:

$$E_k = \begin{bmatrix} 1+0.2\sin(0.1k) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_k = \begin{bmatrix} 0.6-0.1\sin(0.1k) & 0 & 0.3 \\ -0.2\sin(0.2k) & 0.4 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \; B_k = \begin{bmatrix} 1 \\ \sin(0.1k) \\ 0 \end{bmatrix},$$

$$C_k = \begin{bmatrix} 0 & 1+0.2\sin(0.2k) & 0.5 \end{bmatrix}.$$

In the simulation study, the disturbance w_k and measurement noise v_k are bounded as $w_k \in \langle 0, 0.04I_3 \rangle$ and $v_k \in \langle 0, 0.01 \rangle$. They are randomly generated with uniform distributions corresponding to the assumed uncertainty sets. The initial state $x_0 = \begin{bmatrix} 0 & 0 & -0.0301 \end{bmatrix}^{\mathrm{T}}$. We set $\hat{x}_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ and $H_0 = 0.2I_3$. The input is set as $u_k = 0.5\sin(0.15k)$. The maximal zonotope order is chosen as 20. The methods based on Theorem 2 and Theorem 3 are compared in the simulation. For the method based on Theorem 2, we set $S_k = 0$.

The simulation results are shown in Fig. 1. Thanks to sufficient optimization, the estimation results obtained by the method based on Theorem 3 are more accurate than those by the method based on Theorem 2.

To better demonstrate the improvement of Theorem 3, some quantitative results are provided in Table 1, which shows that the method based on Theorem 3 can obtain more accurate estimation bounds for each state.

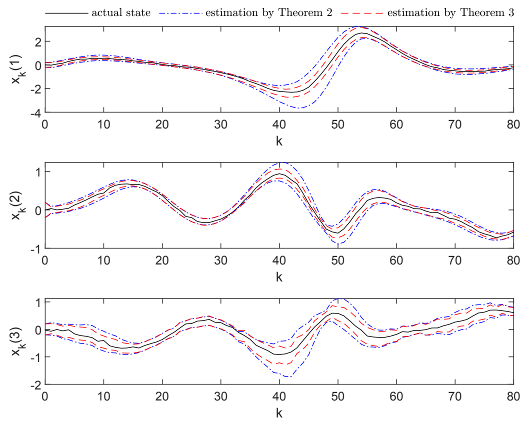

Fig. 1. The interval estimation results of state.

Table 1
The average estimation bounds for each state

Method	$x_k(1)$	$x_k(2)$	$x_k(3)$
Theorem 2	0.5778	0.2440	0.5498
Theorem 3	0.3508	0.1786	0.4024

To more explicitly demonstrate the improvement brought by the sufficient optimization in Theorem 3, the estimation zonotope sets in a few time instants are depicted in Fig. 2, where the green zonotopes are the estimation results obtained by the method based on Theorem 2 and the red ones are those by the method based on Theorem 3. At instant $k = 0$, the two initial zonotopes are identical. They are drawn in gray instead of green or red. Fig. 2 also shows that the method based on Theorem 3 can obtain more accurate estimation results than those by the method based on Theorem 2.

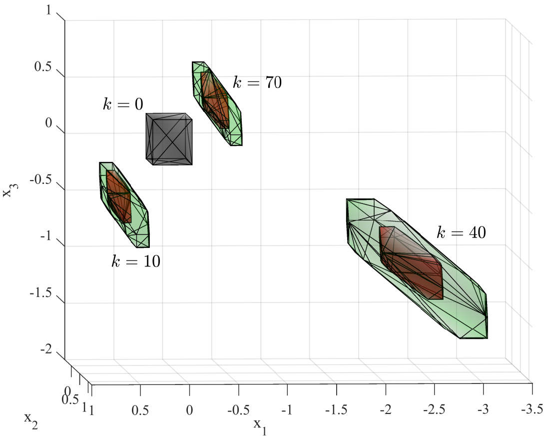

Fig. 2. The estimated zonotopes in a few time instants.

Note that the method in Wang et al. (2019b) cannot apply to the above example since the measurement matrix is time-varying. To better demonstrate the performance of the proposed method based on Theorem 3, an example of a truck-trailer system from Wang et al. (2019b) is used to compare the proposed method with the method in Wang et al. (2019b). The system example in Wang et al. (2019b) is in a polytopic LPV form. It can be reformulated into an equivalent form which is identical to (3). The parameters of the equivalent form are as follows:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_k = \begin{bmatrix} 1 - \frac{\theta_k T_s}{L} & 0 & 0 & 0 \\ \frac{\theta_k T_s}{L} & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -\frac{\theta_k T_s}{2} & 0 & 1 \end{bmatrix},$$

$$B_k = \begin{bmatrix} \frac{\theta_k T_s}{l} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_k = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\theta_k = -0.9 + 0.3\sin(0.0386k + 1)$. And $L = 5.5$m is the length of trailer, $l = 4.8$m is the length of truck, $T_s = 0.2$s is the sampling time.

In the simulation, $x_0 = \begin{bmatrix} -0.1745 & 0.0873 & 3.0 & -0.0057 \end{bmatrix}^{\mathrm{T}}$. We set $H_0 = \mathrm{diag}(\begin{bmatrix} 0.1 & 0.02 & 0.1 & 0.02 \end{bmatrix})$ and $\hat{x}_0 = \begin{bmatrix} -0.1 & 0.08 & 3 & 0 \end{bmatrix}^{\mathrm{T}}$, then $x_0 \in \langle \hat{x}_0, H_0 \rangle$. The disturbance and noise are bounded as $w_k \in \langle 0, D_k \rangle$ and $v_k \in \langle 0, 0.002 I_3 \rangle$, where $D_k = \mathrm{diag}(\begin{bmatrix} 0 & 0 & 0.006\theta_k & 0 \end{bmatrix})$.

The maximal zonotope order is set as 20, which is the same with that in Wang et al. (2019b). w_k and v_k are randomly generated with uniform distributions corresponding to the assumed uncertainty sets. The simulation results are given in Fig. 3, which shows that the estimation results of all state components obtained by the proposed method are more accurate than those by the method in Wang et al. (2019b). Some quantitative results of the accuracy of the two compared methods are given in Table 2, which also illustrates the superiority of the proposed method.

Table 2
The average estimation bounds for each state

Method	$x_k(1)$	$x_k(2)$	$x_k(3)$	$x_k(4)$
Wang et al. (2019b)	0.6606	0.1046	0.0953	0.0397
The proposed one	0.1331	0.0084	0.0063	0.0057

Note that D_k does not satisfy (4), which has been assumed for Theorem 3. Nevertheless, the matrix Σ_k is invertible for all k, as shown by its condition number monitored during the simulation, and the optimality result of Theorem 3 holds also under this weaker condition. See Remark 4 in the previous section. The example with a singular D_k is for the purpose of comparison

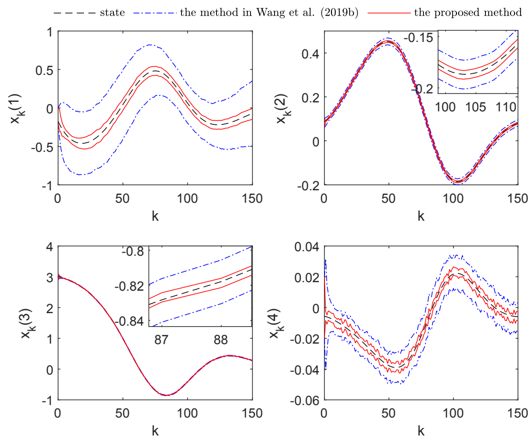

Fig. 3. The estimation results of the two compared methods.

with Wang et al. (2019b), where this example was initially presented. In practice, it is less easy to check the condition on the large matrix Σ_k than the stronger conditions (4), which may be weakened in future studies. To simply rely on the conditions expressed in (4), simulations have also been made after modifications of D_k satisfying (4), leading to visually unnoticeable changes in the numerical results.

6 Conclusion

In this paper, we propose a zonotopic set-membership estimation method for discrete-time LTV descriptor systems. The proposed set-membership estimator is based on combining a single state trajectory estimator with the reachable set of estimation error. We propose an optimization method to design the parameters of the estimator such that the considered size criterion is minimized. The simulation examples have shown the effectiveness of the proposed method.

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